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DYNAMICS OF THE LIQUID CORE OF THE EARTH

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CONTENTS

An asymptotic theory is developed for the long-period bodily tides in an Earth model having a liquid core. The yielding inside the core is found to be different in the case of a stable density stratification from the case of an unstable stratification. In the latter case, a boundary layer is formed in which the stress decreases exponentially with depth below the core surface, the scale length of the exponent being proportional to the frequency. In the limit of vanishing frequency the stress tends to zero through most of the liquid core, except near the boundary layer at the surface, where it grows to a finite value. In case of a stable stratification, the stress oscillates with depth below the surface of the core with a wavelength which is proportional to frequency. An infinite number of 'core oscillations' with indefinitely increasing periods exist in a liquid core with stable stratification, but in the case of an unstable stratification, none exist above the fundamental spheroidal oscillation (53.7 min) for $n = 2$. The assertion made that a liquid core must be in neutral equilibrium is not true. The displacements and stresses within a liquid core in long-period tidal yielding are determinate, even in the static limit, and are not arbitrary. Love numbers are derived for uniformly stable, neutral, and unstable liquid cores, as well as for a model with a rigid inner core.

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1. Introduction

1.1. *The Jeffreys–Vicente conjecture*

In this investigation we aim at clarifying some difficulties which have arisen in the theory of the dynamics of the Earth, stemming from the liquid state of the core. When the theory of the free oscillations of the Earth was formulated (cf. Pekeris & Jarosch 1958; Alterman, Jarosch & Pekeris 1959), where the natural periods concerned were of the order of an hour or less, the fact that for $r < 0.545a$ the core of the Earth has a vanishing rigidity provided a simplification in the analysis rather than a complication. Here a denotes the radius of the Earth, and, in the first instance, we neglect the evidence for the existence of some rigidity in the 'inner core' for $r < 0.196a$. Difficulties appeared when the theory was extended to longer periods, of the order of 12 h, as in the case of the yielding of the Earth in the bodily tide. Jeffreys & Vicente (1966) pointed out, for example, that the values of Love numbers h, k, l published by Alterman *et al.* (1959) for periods of 6 and 12 h and ∞, respectively, are inconsistent with an expected quadratic variation of h with frequency σ of the form

$$h(\sigma) = h^{(0)} + \sigma^2 h^{(2)} + \ldots, \tag{1}$$

and similarly for k and l.

Here $h^{(0)}$ denotes the limiting value of the Love number h as $\sigma \to 0$. This *Jeffreys–Vicente conjecture* is suggested by the form of the differential equations governing the oscillations of the Earth. In a spherical system of coordinates, with origin at the centre of the Earth, let the components of *displacement* (u, v, w) in the directions (r, θ, ϕ) be given by

$$u = U(r)\, S_n(\theta,\phi), \quad v = V(r)\,\frac{\partial S_n(\theta,\phi)}{\partial\theta}, \quad w = \frac{V(r)}{\sin\theta}\,\frac{\partial S_n(\theta,\phi)}{\partial\phi}, \tag{2}$$

$$\Delta = X(r)\, S_n(\theta,\phi), \quad \psi = P(r)\, S_n(\theta,\phi), \tag{3}$$

$$X = \dot U + \frac{2}{r} U - \frac{n(n+1)}{r} V. \tag{4}$$

Here ψ denotes the perturbation in the gravitational potential. The unknown functions $U(r)$, $V(r)$ and $P(r)$ are subject to the differential system of the *sixth order*

$$\sigma^2\rho_0 U + \rho_0 \dot P + g_0\rho_0 X - \rho_0 \frac{d}{dr}(g_0 U) + \frac{d}{dr}(\lambda X + 2\mu\dot U) + \frac{\mu}{r^2}[4\dot U r - 4U + n(n+1)(-U - r\dot V + 3V)] = 0, \tag{5}$$

$$\rho_0\sigma^2 Vr + \rho_0 P - g_0\rho_0 U + \lambda X + r\frac{d}{dr}\left[\mu\left(\dot V - \frac{V}{r} + \frac{U}{r}\right)\right] + \frac{\mu}{r}[5U + 3r\dot V - V - 2n(n+1)V] = 0, \tag{6}$$

$$\ddot P + \frac{2}{r}\dot P - \frac{n(n+1)}{r^2} P = 4\pi G(\dot\rho_0 U + \rho_0 X). \tag{7}$$

Here the dot denotes differentiation with respect to r, $\lambda(r)$ and $\mu(r)$ denote the Lamé constants in the Earth model, and $\rho_0(r)$, $g_0(r)$ the undisturbed values of density and gravity. The dependence on time was assumed to be given by a factor $e^{i\sigma t}$. This system is to be solved subject to the condition of regularity at the origin, and the boundary conditions to be satisfied at the surface $r = a$:

$$y_2 = \lambda X + 2\mu\dot U = \tau_{rr}(a), \tag{8}$$

$$y_4 = \mu\left(\dot V - \frac{V}{r} + \frac{U}{r}\right) = 0, \tag{9}$$

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$$y_6+[(n+1)/a]y_5 = (2n+1)g_0(a), \tag{10}$$

where
$$y_5 = P, \quad y_6 = \dot{P}-4\pi G\rho_0 U, \tag{11}$$

G denoting the gravitational constant, and $\tau_{rr}(a)$ the prescribed applied normal stress in the case of surface loading. In the case of free oscillations, the right-hand sides of (8) and (10) vanish. In the problem of the bodily tide the r.h.s. of (8) vanishes.

The frequency σ enters equations (5) and (6) only through the factor σ^2. Let us denote the differential equations (5) and (6) with the σ^2 terms dropped as the *static differential equations*. A solution ($U^{(0)}$, $V^{(0)}$, $P^{(0)}$) of the static differential system, which satisfied the boundary conditions, will be denoted as the *static solution*. Equations (5) and (6) suggest a solution by a perturbation expansion in the small parameter σ^2 for U, V and P, of the form (1), with the static solution as the leading term. It will be shown in the sequel that this form of an expansion for the core is valid only when the density stratification in the liquid core is one of neutral equilibrium; i.e. when

$$g_0+\lambda\dot{\rho}_0/\rho_0^2 = 0. \tag{12}$$

Equation (12) is known as the Adams–Williamson condition (Adams & Williamson 1923).

When the density stratification does not obey the Adams–Williamson condition, let us define a class of polytropic liquid core models by

$$g_0+\lambda\dot{\rho}_0/\rho_0^2 = \beta(r)\,g_0. \tag{13}$$

A model in which $\beta(r) > 0$ is statically unstable, while in the case of negative $\beta(r)$, the stratification is stable.

We shall designate a model in which $\beta(r)$ is positive throughout the core as *uniformly unstable*, and one in which $\beta(r)$ is negative throughout as *uniformly stable*. It will be helpful to gain insight into the dynamic behaviour of the liquid core if we analyse separately the three classes of uniformly unstable, uniformly stable and uniformly neutral models. The following principle results were found:

(*a*) In the case of uniformly unstable as well as uniformly neutral models, the Jeffreys–Vicente conjecture applies to Love numbers.

(*b*) Uniformly unstable and uniformly neutral models have no free oscillations with periods greater than about 53.7 min, which is the period of the fundamental spheroidal oscillation for $n = 2$.

(*c*) Uniformly stable models have an unlimited number of 'core oscillations' with periods ranging from the fundamental of about 53.7 min to ∞.

(*d*) In the case of uniformly stable models, the Love numbers show a nearly linear variation with σ^2, if we exclude the regions near the resonances.

1.2. *The static solution in the case of a liquid core*

That the liquidity of the core introduces peculiar difficulties in the static limit was already noted when the equations were formulated in 1956. Putting the rigidity μ equal to zero in equations (5) and (6), we get

$$\sigma^2\rho_0 U+\rho_0\dot{P}+g_0\rho_0 X-\rho_0\frac{\mathrm{d}}{\mathrm{d}r}(g_0 U)+\frac{\mathrm{d}}{\mathrm{d}r}(\lambda X) = 0, \tag{14}$$

$$\sigma^2\rho_0 Vr+\rho_0 P-\rho_0 g_0 U+\lambda X = 0. \tag{15}$$

Differentiating (15) with respect to r and subtracting (14) from the result, we get

$$\sigma^2[\mathrm{d}(\rho_0 Vr)/\mathrm{d}r-\rho_0 U]+\dot{\rho}_0(P-g_0 U)-g_0\rho_0 X = 0, \tag{16}$$

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which, by (15), becomes

$$(g_0+\lambda\dot{\rho}_0/\rho_0^2)X = \sigma^2(V+r\dot{V}-U). \tag{17}$$

In the static limit of $\sigma^2 = 0$, equation (17) requires that either

$$\text{I:}\quad g_0+\lambda\dot{\rho}_0/\rho_0^2 = 0, \tag{18}$$

or

$$\text{II:}\quad X = 0, \tag{19}$$

unless the term $(V+r\dot{V}-U)$ grows indefinitely. The notes of 1956 (unpublished) read:

> 'In case I the internal motion inside the core is indeterminate since you can have steady circulation inside an adiabatic medium, provided the motion does not reach the boundaries.'

This is not helpful, since the motion does reach the boundaries.

Conditions (18) and (19) were first published by Longman (1963), who pointed out that if condition (19) is assumed to hold in the static limit, then it is impossible to satisfy the boundary conditions. Longman therefore concluded that:

(*e*) It is mandatory that the Adams–Williamson condition (18) be satisfied in the liquid core.

As a corollary of dictum (*e*), Longman further concluded that:

(*f*) A uniform liquid core of finite compressibility is physically impossible because the first term in (18) would be positive and the second would be zero.

1.3. *The uniform liquid sphere*

That dictum (*f*) is not valid can be demonstrated readily by considering the tidal yielding of an Earth model consisting entirely of a *liquid sphere of uniform density*. In such a model

$$g_0(r) = Ar, \quad A = \tfrac{4}{3}\pi G\rho_0. \tag{20}$$

An exact solution of equations (7), (14) and (15) which satisfies the boundary conditions (8), (9) and (10) in the case of the bodily tide for $n = 2$ is

$$U = Br, \quad V = \tfrac{1}{2}Br, \quad X = 0, \tag{21}$$

$$P = B(A-\tfrac{1}{2}\sigma^2)r^2, \quad B = \frac{5}{2-(5\sigma^2/2A)}. \tag{22}$$

The Love numbers are

$$h = \frac{U(a)}{a} = \frac{5}{2-(5\sigma^2/2A)}, \tag{23}$$

$$k = \{[P(a)/Aa^2]-1\} = \frac{3}{2-(5\sigma^2/2A)}, \tag{24}$$

$$l = \tfrac{1}{2}h. \tag{25}$$

Resonance occurs when

$$\sigma^2 = \tfrac{4}{5}A = \tfrac{4}{5}\frac{g_0(a)}{a}. \tag{26}$$

Equation (26) is in agreement with Kelvin's result (Lamb 1932):

$$\sigma_n^2 = \frac{2n(n-1)g}{(2n+1)a}, \tag{27}$$

which was derived for an incompressible fluid.

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Clearly, we have here a model which violates the Adams–Williamson condition (18), in contradiction to propositions (*e*) and (*f*), and yet responds to the tidal potential in a simple manner in the whole range of frequencies starting with the static case and continuing up to the resonance frequency.

1.4. *The liquid core and the Adams–Williamson condition*

That the Adams–Williamson condition cannot be mandatory for self-gravitating fluids is exemplified by the structure of our own troposphere, where the temperature gradient is only about 0.6 of the adiabatic value. If the condition of neutral equilibrium were mandatory, then the theory of stellar models would be considerably simplified; 'as it isn't, it ain't'.

Longman's proposition (*f*) was adopted by Dahlen (1971 *a*) in his investigation of the excitation of the Chandler wobble by earthquakes. Smylie & Mansinha (1971) reject the Longman proposition and get different results for the static solution. They, however, replace the adiabatic condition (18) by one allowing for *a discontinuity in the radial component of displacement at the core boundary*. This is objected to by Dahlen (1971 *b*), since it invokes cavitation.

The problem of the static deformation of the Earth has come to the fore recently through the work of Press (1965), who succeeded in measuring the change of the level of residual strain by earthquakes at teleseismic distances. Our investigation shows that for periods of the order of the bodily tide and longer, the yielding of the Earth depends on the structure of the liquid core. Whether these low-frequency deformations will supply evidence which may allow us to discriminate between a density distribution in a liquid core which is stable ($\beta < 0$) and one which is unstable ($\beta > 0$) is at present doubtful. In the unstable case we find that the stress y_2 diminishes exponentially with depth below the core surface. The depth-scale of the exponential drop is proportional to σ, so that in the limit of $\sigma \to 0$ the stress tends to become discontinuous at the core boundary.

1.5. *The boundary layer near the surface of the liquid core*

Our resolution of the difficulty with the static limit in the case of a liquid core is as follows:

(*g*) The dynamics of the liquid core of the Earth do not impose any restriction on its density stratification.

(*h*) In the case of a uniformly unstable stratification ($\beta > 0$) the stress, as well as the divergence of the displacement X, tend to zero with diminishing σ throughout the liquid core, except for a boundary layer of diminishing thickness near the core surface. Within the boundary layer the stress rises steeply from a near-zero value to a finite value.

The stress referred to here and elsewhere is the radial one, since in a liquid the transverse components of stress are zero anyhow.

This result implies that in the case $\beta > 0$, the condition (19) is approached through most of the core as $\sigma \to 0$. The variation of stress within the boundary layer has a dynamic effect, so that, in the static limit, the jump in stress at the core boundary is not as arbitrary as is the discontinuity in tangential displacement V.

(*k*) In the case of a uniformly stable stratification ($\beta < 0$) the stress has a term which oscillates with depth below the core boundary, the depth scale varying like σ. The stress and the divergence of displacement X do not tend to zero, because of the excitation of the free core oscillations.

(*l*) In all cases, including one of neutral equilibrium, the yielding inside the core in the static limit is determinate, and not arbitrary.

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1.6. *The boundary conditions for the static solution*

When $\sigma \to 0$, equation (15) takes on the form

$$P - g_0 U = -(\lambda/\rho_0) X. \tag{28}$$

If, in addition, we assume that $X = 0$, we have

$$y_2 = \tau_{rr} = \lambda X = 0, \tag{29}$$

$$U = P/g_0, \quad V = \frac{1}{n(n+1)}(r\dot{U} + 2U), \tag{30}$$

so that both U and V are determined from P, the latter being the solution of equation (7) which is finite at the origin. The difficulty with the static solution was that the amplitude of the solution of (7), together with the allowed discontinuity of V at the core boundary, gave only two arbitrary constants out of the three needed to satisfy the three boundary conditions (8), (9) and (10) at the surface $r = a$. The resolution of this difficulty is provided by the existence of the boundary layer within which X changes very rapidly to the value at the top of the surface of the core. Equation (28) shows that at the top of the boundary of the core we must have

$$P - g_0 U + y_2/\rho_0^c = 0, \quad r = b, \tag{31}$$

where y_2 is the stress defined in (8), ρ_0^c is the value of the density at the surface of the core, and b denotes the radius of the core. Since P, U and y_2 are continuous at the core-mantle interface, equation (31) provides a condition on the mantle solutions.

Let $U^{(0)}$, $V^{(0)}$, $P^{(0)}$ be the 'static solution' in the core obtained by putting

$$\sigma^2 = 0, \quad X = 0, \tag{32}$$

solving (7) for P, and from it for U and V by (30).

Let
$$\gamma(r) = y_6^{(0)}/y_5^{(0)} = [\dot{P}^{(0)} - 4\pi G \rho_0^c U^{(0)}]/P^{(0)}, \tag{33}$$

then the following relation is found to hold for the mantle solutions:

$$\dot{P} - 4\pi G \rho_0^m U - \gamma(b) P + 4\pi G y_2/g_0 = 0, \quad r = b, \tag{34}$$

where ρ_0^m is the density at the bottom of the mantle.

A knowledge of the constant $\gamma(b)$ is all that is needed for the static solution in the mantle. Equations (5) and (6), with the σ^2 terms dropped, together with (7), constitute a differential system of the sixth order, for which six boundary conditions are required. Three are given at the surface by (8), (9) and (10). The remaining three are given by (31) and (34) and by the condition

$$y_4 = \mu\left(\dot{V} - \frac{V}{r} + \frac{U}{r}\right) = 0, \quad r = b. \tag{35}$$

2. Dynamics of an Earth model consisting of a uniform liquid core enclosed by a uniform solid mantle

The dynamical effects of the liquidity of the core at vanishing frequencies are so complex that it is advisable to present the analysis first for a simplified model before proceeding to the general case of a model with continuously varying properties. We shall discuss the 'α-model' which was

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treated by Alterman *et al.* (1959). It consists of an homogeneous solid mantle of constant parameters ρ_1, λ_1 and μ_1 enclosing a liquid core with constant ρ_0 and λ_0. When these constants were adjusted so as to equal the average values in the respective regions of Bullen's model B, the fundamental period for $n = 2$ came out 56.0 min, as against the value of 53.7 for model Bullen B.

With ρ_0 in the core constant, put

$$\tfrac{4}{3}\pi G\rho_0 = A, \quad \alpha = \sigma^2/A, \quad Q = P/A, \tag{36}$$

$$I_1(r) = \frac{1}{r^{2+\alpha}}\int_0^r r^{1+\alpha} V \,\mathrm{d}r. \tag{37}$$

Then within the core we have

$$g_0(r) = Ar, \tag{38}$$

and (17) becomes

$$rX = \alpha(V + r\dot{V} - U) = r\dot{U} + 2U - n(n+1)V, \tag{39}$$

by (4). Equations (7) and (15) now read

$$\ddot{Q} + \frac{2}{r}\dot{Q} - \frac{n(n+1)}{r^2}Q = 3X, \tag{40}$$

$$(\lambda_0/\rho_0 A)X = rU - \alpha rV - Q. \tag{41}$$

The solution of (39) and (40) is

$$U = \alpha V + (n-\alpha)(n+\alpha+1)I_1(r), \tag{42}$$

$$Q = Br^n + 3\alpha r I_1(r). \tag{43}$$

Substituting (42) and (43) in (41), we get

$$(\lambda_0/\rho_0 A)X = -Q + (n-\alpha)(n+\alpha+1)(Q - Br^n)/3\alpha, \tag{44}$$

by which (40) takes on the form

$$\frac{\mathrm{d}^2Q}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}Q}{\mathrm{d}r} - \frac{n(n+1)}{r^2}Q + \frac{\rho_0 A}{\lambda_0}\left[4 + \alpha - \frac{n(n+1)}{\alpha}\right]Q = Kr^n. \tag{45}$$

Transforming to the non-dimensional variable

$$s = r/a, \tag{46}$$

we get

$$\frac{\mathrm{d}^2Q}{\mathrm{d}s^2} + \frac{2}{s}\frac{\mathrm{d}Q}{\mathrm{d}s} - \frac{n(n+1)}{s^2}Q + \frac{\rho_0 Aa^2}{\lambda_0}\left[4 + \alpha - \frac{n(n+1)}{\alpha}\right]Q = Ls^n. \tag{47}$$

Let

$$Q = Ds^n + T, \quad D = L\alpha\lambda_0/[\rho_0 Aa^2(4\alpha + \alpha^2 - n - n^2)]; \tag{48}$$

then T satisfies the homogeneous equation

$$\frac{\mathrm{d}^2T}{\mathrm{d}s^2} + \frac{2}{s}\frac{\mathrm{d}T}{\mathrm{d}s} - \frac{n(n+1)}{s^2}T + \frac{\rho_0 Aa^2}{\lambda_0}\left[4 + \alpha - \frac{n(n+1)}{\alpha}\right]T = 0. \tag{49}$$

As $\sigma^2 \to 0$, and with it α, the dominant terms in (49) become the first and the last, leading to the asymptotic form

$$T \simeq \frac{F}{s}\mathrm{e}^{\nu s} + O(\alpha), \tag{50}$$

with

$$\nu^2 = \rho_0 Aa^2 n(n+1)/\lambda_0\alpha, \quad \nu = (aA/\sigma)[n(n+1)\rho_0/\lambda_0]^{\frac{1}{2}}. \tag{51}$$

Asymptotically, the solution T varies like $\exp(\sigma^{-1})$, and not like the form (1), underlying the Jeffreys–Vicente conjecture.

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The solution inside the liquid core consists, like the expression for Q in (48), of a regular part plus a boundary layer term of the form (50). In the latter we retain only the leading asymptotic term. With

$$s_0 = b/a, \tag{52}$$

and $\bar{\rho}$ denoting the mean density of the Earth, we have

$$U/a = Z_1 = Es^{n-1} + \frac{n(n+1)F}{s^2} e^{\nu(s-s_0)}, \tag{53}$$

$$V/a = Z_3 = \frac{E}{n} s^{n-1} + \frac{\nu F}{s} e^{\nu(s-s_0)}, \tag{54}$$

$$P/ag_0(a) = Z_5 = (\rho_0/\bar{\rho}) \left[Eds^n + \frac{3\alpha F}{s} e^{\nu(s-s_0)} \right], \tag{55}$$

$$y_6/g_0(a) = Z_6 = (\rho_0/\bar{\rho}) \left[E(nd-3) s^{n-1} - \frac{3n(n+1)F}{s^2} e^{\nu(s-s_0)} \right], \tag{56}$$

$$Z_2 = X = \frac{\alpha \nu^2 F}{s} e^{\nu(s-s_0)}, \quad d = [1 - \sigma^2/nA]. \tag{57}$$

The two arbitrary constants E and F, together with the jump in V at the core surface, provide the three parameters needed to satisfy the three boundary conditions at the surface of the Earth.

We note that the stress function Z_2 contains only the boundary-layer term which drops with depth below the core surface as $\exp[(r-r_0)/L]$, where

$$L = \frac{a}{\nu} = \frac{3\sigma}{4\pi a G\rho_0[n(n+1)\rho_0/\lambda_0]^{\frac{1}{2}}} = \frac{90}{T}, \tag{58}$$

for the constants of the α-model. Here L is expressed in kilometres and the period T in days. Thus, in the case of the fortnightly tide, the stress drops to a fraction 1/e at a depth of 6 km below the surface of the core.

We have solved for the Love numbers ($n = 2$) in the α-model for periods of 2, 4, 6, 12 and 24 h, using first the exact equations (5), (6) and (7), or the system of first-order differential equations corresponding to them. The solutions were started by power-series expansions near the origin, and then continued numerically by the Runge–Kutta method. The Love numbers are given in table 1. The corresponding stress function $Z_2(s)$ is plotted in figure 1. An approximate solution was also obtained by using, for the region inside the core, the asymptotic representations given in (53) to (57). The asymptotic solutions are shown in figure 1 by the dotted lines. It is seen that already at a period of 6 h the asymptotic solution gives a close approximation to the exact solution, thus substantiating the reality of the boundary layer.

In any case, we have here a model which has a structure violating (18), and which shows a tendency to approach condition (19) through most of the core as $\sigma \to 0$.

It is seen from figure 1 that as $\sigma \to 0$, Z_2 nearly vanishes throughout most of the liquid core, except for a layer of decreasing thickness near the core boundary. One might be tempted to assume that the boundary condition in the static limit is the vanishing of X in the liquid core, coupled with a discontinuous jump in X at the bottom of the mantle. This jump in X, together with the discontinuity in V, would allow for the satisfaction of the boundary conditions at the Earth's surface. A solution based on this assumption was carried out, and is designated as 'static' in table 1. It is seen that the 'static' values deviate appreciably from the limit indicated by the

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trend of the Love numbers up to a period of 24 h. If we take, for example, the values given for $T = 12$ and 24 h, and extrapolate to $T = \infty$ by equation (1), we get

$$h = 0.6904, \quad k = 0.3535, \quad l = 0.1145. \tag{59}$$

The value of Z_2 at the bottom of the mantle comes out 0.163 in the 'static solution' as against the value of 0.070 which holds in the exact solution for all periods greater than 3 h.

The above results indicate that the boundary layer has a finite dynamic effect even though its thickness vanishes in the limit. The proper boundary conditions to be applied at the core surface can be deduced by examining the limiting form assumed by the asymptotic equations (53) to (57) as $\sigma \to 0$. First, the factor $e^{\nu(s-s_0)}$ approaches unity, giving, in place of (53),

$$Z_1 = Es_0^{n-1} + n(n+1)\,F/s_0^2. \tag{60}$$

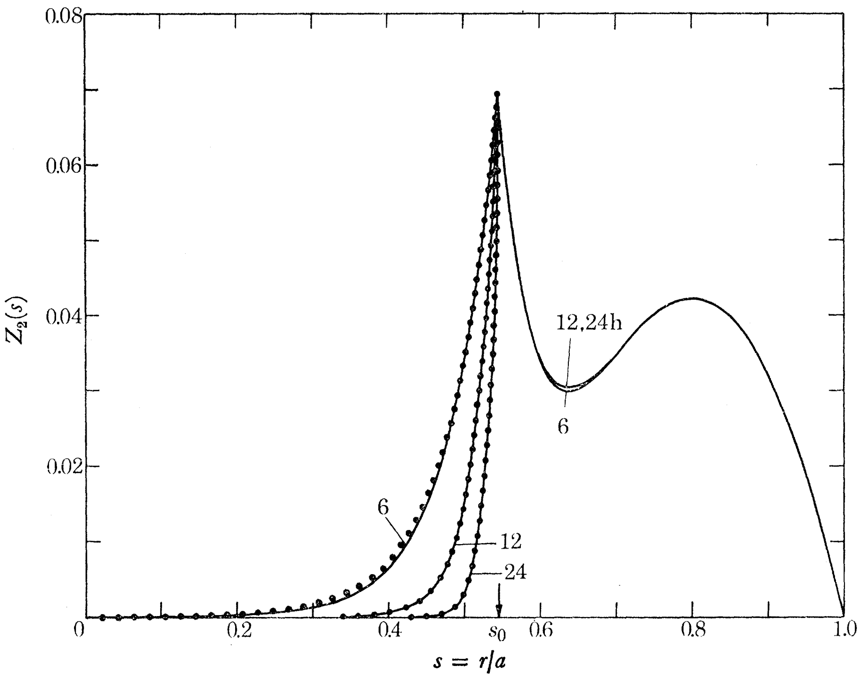

FIGURE 1. The function $Z_2(s)$ for the 'α-model' for periods of 6, 12 and 24 h. —, exact solution; •••, asymptotic solution. $n = 2$.

TABLE 1. LOVE NUMBERS h, k, l FOR THE 'α-MODEL' AS FUNCTIONS OF THE TIDAL PERIOD T

The 'static' values were obtained by assuming the stress Z_2 to vanish inside the core and to be discontinuous at $s = s_0$. $h^{(0)}$, $k^{(0)}$ and $l^{(0)}$ result from using the asymptotic equations (60) to (63). $h^{(2)}$, etc., are defined in equation (1).

T/hour		h	k	l
2	—	0.8766	0.4463	0.1320
4	—	0.7296	0.3728	0.1182
6	—	0.7076	0.3619	0.1161
12	—	0.6948	0.3556	0.1149
18	—	0.6924	0.3544	0.1147
24	—	0.6915	0.3540	0.1146
∞	$h^{(0)}, k^{(0)}, l^{(0)}$	0.6903	0.3535	0.1145
	$h^{(2)}, k^{(2)}, l^{(2)}$	2.403×10^5	1.014×10^5	1.65×10^4
	static	0.5844	0.2983	0.1118

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Equation (54) is immaterial, since V is discontinuous at $s = s_0$. The remaining equations become

$$Z_5 = (\rho_0/\bar{\rho})Eds_0^n, \tag{61}$$

$$Z_6 = (\rho_0/\bar{\rho})\,[E(nd-3)\,s_0^{n-1} - 3n(n+1)F/s_0^2], \tag{62}$$

$$Z_2 = \alpha\nu^2 F/s_0 = \rho_0 Aa^2 n(n+1)F/\lambda_0 s_0. \tag{63}$$

These equations retain the two arbitrary constants E and F needed in order to satisfy the boundary conditions at the surface of the Earth.

The results of the solution obtained by using equations (60) to (63) in the core, with $\sigma = 0$, are given in table 1 under $T = \infty$, and are designated as $h^{(0)}$, $k^{(0)}$ and $l^{(0)}$. Using these values of $h^{(0)}$, $k^{(0)}$ and $l^{(0)}$ and the Love numbers for $T = 24$ h in equation (1), we get the values of $h^{(2)}$, $k^{(2)}$ and $l^{(2)}$. The resulting formulae reproduce the Love numbers to better than 1 % down to periods of 4 h.

In order further to substantiate the existence of the boundary layer, we have carried out a solution by assuming Z_2 to be zero for $s < s' = s_0 - \delta$, and then allowing for a jump in Z_2 at s'. The required jump in Z_2 came out very small. Thus in the case of 24 h and $\delta = 2/\nu$, corresponding to $s' = 0.517$ (s_0 being 0.545), the required jump in Z_2 came out only -2.16×10^{-4} as against 0.07 at the core boundary, and the resulting values of the Love numbers agreed with the exact values to within 10^{-4}.

According to Jeffreys & Vicente (1966), the relative difference between the static values of the Love numbers and the values at a low frequency σ should be of the order of σ^2/σ_0^2, σ_0 denoting the fundamental natural frequency of around 53.7 min ($= 1.9 \times 10^{-3}\,\mathrm{s}^{-1}$). According to equation (1), this implies that $h^{(2)}/h^{(0)} \simeq 1/\sigma_0^2 = 2.8 \times 10^5$. It is seen from table 1 that this order of magnitude is substantiated.

It is of interest to apply the solution obtained in this section for a homogeneous core to the solution for a uniform sphere given in the previous section. When the core boundary reaches to the surface $r = a$, then $s_0 = 1$ and $\rho_0 = \bar{\rho}$. Applying the boundary conditions $Z_2 = 0$ (equation (8)) at $s_0 = 1$ and using (63) we get $F = 0$: there is no boundary layer in the case of the bodily tide in a uniform liquid sphere. With the boundary-layer terms dropped in (60) and (62), we get from the boundary condition (10) the Love numbers given in equations (23), (24) and (25). The latter were derived for a finite value of the Lamé constant λ, and since the divergence X was taken to vanish, the stress $y_2 = \lambda X$ vanishes throughout the sphere. In Kelvin's treatment (Lamb 1932) the liquid is stated to be incompressible. This means that $X = 0$, but that λX may be finite if $\lambda = \infty$. Our solution shows that in order to satisfy the boundary conditions, $\lambda X\,(= -p)$ must vanish throughout the sphere.

3. Polytropic models of the liquid core of the Earth

When we came to explore the yielding, at low frequencies, of realistic Earth models, the results turned out to be more complicated than in the case of the uniform core shown in figure 1. In the case of the 'Gutenberg' model, for example (Dorman, Ewing & Oliver 1960), we found that one source of complication was the fact that the density stratification changes from a stable one in the inner part of the core to an unstable one in the outer part. We found that the two classes of liquid core models, with $\beta(r)$ in (13) positive or negative respectively, exhibit different types of yielding at low frequencies. We shall therefore discuss separately uniformly stable models with $\beta(r) < 0$, on the one hand, and uniformly unstable models with $\beta(r) > 0$, on the other hand.

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As a basis, we adopted the model M_3 of Landesman, Satô & Nafe (1965), as modified by Pekeris (1966). The modification is minor throughout the core and reaches a maximum of only 4.6 % in the mantle. Leaving the density distribution in the mantle as given, we now modified the density distribution within the liquid core so as to keep $\beta(r)$ in (13) constant throughout. This was obtained by solving the simultaneous differential equations

$$\frac{d\rho_0}{dr} = -\frac{\rho_0}{c_P^2}(1-\beta)g_0, \tag{64}$$

$$\frac{dg_0}{dr} = -\frac{2}{r}g_0 + 4\pi G\rho_0. \tag{65}$$

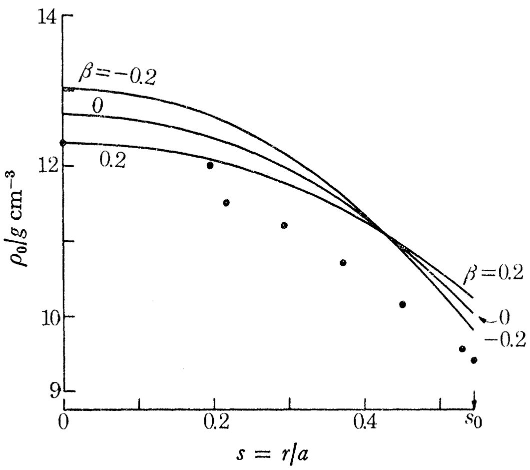

Figure 2. Distribution of density $\rho_0(s)$ in the liquid core for uniform polytropic models $\beta = -0.2$, 0, 0.2 (equation 13). •, Gutenberg model.

The solutions are constrained by the fixed value of the total mass of the Earth:

$$M = 5.977 \times 10^{27}\,\mathrm{g}, \tag{66}$$

and by the moment-condition

$$I_2 = \tfrac{8}{3}\pi \int_0^a \rho_0(r) r^4\,dr = 0.330841 Ma^2. \tag{67}$$

I_2 is insensitive to the small changes introduced in the core, so that, effectively, the starting value $\rho_0(0)$ is constrained by condition (66) only. Figure 2 and table 2 show the density distributions $\rho_0(s)$ within the core obtained for the cases $\beta = -0.2$, 0 and 0.2.

The $\beta = 0$ model is close to the original M_3 model, while the others deviate from it by less than 3 %. It is seen from table 3 that the residuals between the theoretical periods of free spheroidal oscillation T_n^c and the observed values T_n^o are of the order of magnitude as for model M_3.

4. Asymptotic theory of long-period bodily tides for an Earth model having a liquid core with continuously varying properties

In the case of a liquid core with continuously varying properties, the dynamical equations to be solved are (7), (15) and (17). With

$$\beta(r) = (g_0 + \lambda_0\dot\rho_0/\rho_0^2)/g_0, \quad \alpha(r) = r\sigma^2/\beta g_0, \tag{68}$$

$$\tau(r) = \int_0^r (\alpha/r)\,dr, \quad I_3 = (1/r^3)\,e^{-\tau}\int_0^r e^{\tau} r V(n^2+n-\alpha-\dot\alpha r-\alpha^2)\,dr, \tag{69}$$

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TABLE 2. EARTH MODEL M_3 (PEKERIS 1966) WITH UNIFORM POLYTROPIC CORES FOR $\beta = -0.2$, 0, 0.2 (SEE EQUATION (13))

model			M_3	$\beta = -0.2$	$\beta = 0$	$\beta = 0.2$
r/km	c_P/km s^{-1}	c_S/km s^{-1}	ρ_0/g cm^{-3}	ρ_0/g cm^{-3}	ρ_0/g cm^{-3}	ρ_0/g cm^{-3}
6371	6.30	3.55	2.840	—	—	—
6338	6.30	3.55	2.840	—	—	—
6338	8.16	4.65	3.386	—	—	—
6311	8.15	4.60	3.474	—	—	—
6271	8.00	4.40	3.488	—	—	—
6221	7.85	4.35	3.462	—	—	—
6171	8.05	4.40	3.413	—	—	—
6071	8.50	4.60	3.374	—	—	—
5958	9.06	5.00	3.569	—	—	—
5871	9.60	5.30	3.812	—	—	—
5771	10.10	5.60	4.047	—	—	—
5671	10.50	5.90	4.215	—	—	—
5571	10.90	6.15	4.373	—	—	—
5471	11.30	6.30	4.502	—	—	—
5371	11.40	6.35	4.619	—	—	—
5171	11.80	6.50	4.852	—	—	—
4971	12.05	6.60	4.955	—	—	—
4771	12.30	6.75	5.040	—	—	—
4571	12.55	6.85	5.066	—	—	—
4371	12.80	6.95	5.072	—	—	—
4171	13.00	7.00	5.085	—	—	—
3971	13.20	7.10	5.090	—	—	—
3771	13.45	7.20	5.092	—	—	—
3571	13.70	7.25	5.086	—	—	—
3491	13.70	7.20	5.239	—	—	—
3473	13.65	7.20	5.279	—	—	—
3473	8.04	—	10.087	9.795	10.020	10.224
3123	8.44	—	10.637	10.449	10.573	10.671
2776	8.90	—	11.082	11.023	11.051	11.051
2429	9.31	—	11.478	11.517	11.457	11.370
2082	9.63	—	11.809	11.939	11.799	11.635
1735	9.88	—	12.079	12.293	12.084	11.854
1388	10.08	—	12.290	12.581	12.314	12.030
1318.6	10.11	—	12.321	12.630	12.354	12.060
1297.8	10.11	—	12.330	12.645	12.365	12.069
1283.9	10.17	—	12.337	12.654	12.373	12.075
1249.2	10.48	—	12.352	12.677	12.390	12.088
1214.5	10.76	—	12.368	12.697	12.407	12.101
1179.8	10.93	—	12.382	12.717	12.422	12.113
1145.1	11.04	—	12.400	12.735	12.437	12.124
1110.4	11.09	—	12.412	12.753	12.451	12.134
1075.7	11.12	—	12.429	12.770	12.464	12.144
1041.0	11.13	—	12.443	12.786	12.477	12.154
867.5	11.15	—	12.501	12.860	12.536	12.199
694.0	11.17	—	12.551	12.921	12.584	12.235
520.5	11.17	—	12.590	12.968	12.621	12.263
347.0	11.16	—	12.614	13.003	12.648	12.284
173.5	11.15	—	12.629	13.023	12.665	12.296
0	11.15	—	12.635	13.030	12.670	12.300

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TABLE 3. THE RESIDUALS $T_n^o - T_n^c$ BETWEEN THE OBSERVED PERIODS OF SPHEROIDAL OSCILLATIONS T_n^o AND THE COMPUTED PERIODS T_n^c FOR MODEL M_3 AND FOR THE POLYTROPIC MODELS $\beta = -0.2$, 0 AND 0.2

	model	M_3	$\beta = -0.2$	$\beta = 0$	$\beta = 0.2$
	T_n^o	$T_n^o - T_n^c$	$T_n^o - T_n^c$	$T_n^o - T_n^c$	$T_n^o - T_n^c$
n	s	s	s	s	s
2	3233.1	6.9	8.1	5.1	1.5
3	2139.2	4.4	4.6	2.9	1.1
4	1546.0	0.1	0.4	−1.0	−2.3
5	1188.4	−1.7	−1.3	−2.4	−3.4
6	962.3	−1.0	−0.7	−1.5	−2.3
7	809.1	−2.8	−2.6	−3.2	−3.7
8	707.7	0.2	0.3	0.0	−0.3
9	634.0	0.6	0.5	0.4	0.2
10	579.3	0.4	0.3	0.2	0.2
11	536.8	0.1	0.1	0.0	0.0
12	502.3	0.1	0.0	0.0	0.0
13	473.2	0.0	0.0	0.0	0.0
14	448.4	0.3	0.2	0.2	0.2
15	426.3	0.0	0.0	0.0	0.0
16	406.8	−0.1	−0.2	−0.2	−0.2

(17) becomes

$$rX = \alpha(V + r\dot{V} - U) = r\dot{U} + 2U - n(n+1)V, \tag{70}$$

of which the solution is

$$U = \alpha V + rI_3. \tag{71}$$

Using (15) and (71), we get

$$\dot{\rho}_0 U + \rho_0 X = (\rho_0^2/\lambda_0)[\beta g_0(U - \alpha V) - P] = (\rho_0^2/\lambda_0)(\beta g_0 r I_3 - P). \tag{72}$$

Let

$$Q = r^2 \frac{d^2P}{dr^2} + 2r\frac{dP}{dr} - n(n+1)P + 4\pi G\frac{\rho_0^2 r^2}{\lambda_0}P; \tag{73}$$

then (7) reads

$$Q = 4\pi G(\rho_0^2/\lambda_0)\beta g_0 r^3 I_3, \tag{74}$$

from which it follows that

$$\begin{aligned} d(\lambda_0 Q e^{\tau}/\rho_0^2\beta g_0)/dr &= 4\pi G e^{\tau} r V(n^2 + n - \alpha - \alpha^2 - \dot{\alpha} r) \\ &= \lambda_0 Q \alpha e^{\tau}/r\rho_0^2\beta g_0 + e^{\tau} d(\lambda_0 Q/\rho_0^2\beta g_0)/dr. \end{aligned} \tag{75}$$

Hence

$$d(\lambda_0 Q/\rho_0^2\beta g_0)/dr = 4\pi G n(n+1) r V + O(\alpha V, \alpha Q). \tag{76}$$

We now express the left-hand side of (76) in terms of V. Eliminating U between (15), (70) and (71), we get

$$P + \frac{\alpha V\lambda_0}{\rho_0}\left[\frac{\dot{\rho}_0}{\rho_0} + \frac{(1-\alpha)}{r}\right] + \frac{\alpha\lambda_0}{\rho_0}\dot{V} = \left(g_0 + \frac{\alpha\lambda_0}{\rho_0 r}\right) r I_3, \tag{77}$$

which, by (74), gives

$$\frac{\lambda_0 Q}{\rho_0^2\beta g_0} = \frac{4\pi G}{(g_0 + [\alpha\lambda/\rho_0 r])}\left[r^2 P + \frac{\alpha r\lambda_0}{\rho_0}\frac{d}{dr}(rV) + \frac{\alpha r\lambda_0}{\rho_0}V\left(\frac{r\dot{\rho}_0}{\rho_0} - \alpha\right)\right]. \tag{78}$$

Let

$$c^2(r) = \frac{(n^2+n)g_0\rho_0}{\lambda_0 r\alpha} = \frac{(n^2+n)\rho_0\beta g_0^2}{\lambda_0 r^2\sigma^2} \equiv \frac{\nu^2(s)}{a^2}, \tag{79}$$

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$c(r)$ being a large parameter for decreasing σ, which becomes imaginary for stable density stratifications, when $\beta < 0$. We expect, subject to *a posteriori* verification, that, relative to the middle term in brackets in (78), the first term is of order c^{-2} while the last term is of order c^{-1}. Neglecting these, we get from (76)

$$\frac{d}{dr}\left[\frac{\alpha r\lambda_0}{\rho_0(g_0+\alpha\lambda/\rho_0 r)}\frac{d}{dr}(rV)\right] = n(n+1)rV \simeq \frac{\alpha r\lambda_0}{\rho_0 g_0}\frac{d^2}{dr^2}(rV)+O(c^{-1}), \tag{80}$$

or

$$d^2(rV)dr^2 = c^2 rV, \tag{81}$$

$$rV = A\exp\left\{-\int_r^b c(r)\,dr\right\}+O(c^{-1}). \tag{82}$$

To within terms of order c^{-1} we also have

$$e^{-\tau}\int_0^r e^{\tau} rV(n^2+n-\alpha-\dot{\alpha}r-\alpha^2)\,dr \simeq [(n^2+n)A/c]\exp\left\{-\int_r^b c\,dr\right\}; \tag{83}$$

hence, by (71),

$$U \simeq \frac{A(n+n^2)}{r^2 c}\exp\left\{-\int_r^b c\,dr\right\}. \tag{84}$$

It follows now from (70) that

$$X \simeq \alpha Vc = (\alpha c/r)A\exp\left\{-\int_r^b c\,dr\right\}. \tag{85}$$

Since, by (73), $Q \simeq r^2c^2P$, we get from (74) and (83)

$$P = \frac{4\pi GA(n+n^2)\rho_0^2\beta g_0}{c^3 r^2\lambda_0}\exp\left\{-\int_r^b c\,dr\right\} = O(c^{-3}). \tag{86}$$

Let $U^{(0)}$, $V^{(0)}$ and $P^{(0)}$ be the solutions of equations (7), (15) and (17) obtained by putting both σ and X equal to zero; i.e.

$$\ddot{P}^{(0)}+\frac{2}{r}\dot{P}^{(0)}-\frac{n(n+1)}{r^2}P^{(0)} = 4\pi G\dot{\rho}_0 U^{(0)}, \tag{87}$$

$$U^{(0)} = P^{(0)}/g_0, \quad X^{(0)} = 0, \tag{88}$$

$$V^{(0)} = [r\dot{U}^{(0)}+2U^{(0)}]/(n+n^2). \tag{89}$$

Then, with

$$\nu(s) = (g_0/\sigma s)\,[(n^2+n)\beta\rho_0/\lambda_0]^{\frac{1}{2}}, \quad M(s) = \exp\left(-\int_s^{s_0}\nu(s)\,ds\right), \tag{90}$$

we have

$$U = EU^{(0)}+FM/\nu s^2, \tag{91}$$

$$V = EV^{(0)}+FM/s(n^2+n), \tag{92}$$

$$P = EP^{(0)}, \tag{93}$$

$$X = (\rho_0 g_0/\lambda_0\nu)FM/s^2, \tag{94}$$

$$y_6 = Ey_6^{(0)}-(4\pi G/g_0)\lambda_0 X. \tag{95}$$

As in the case of the 'α-model', the two arbitrary constants E and F suffice to satisfy the boundary conditions.

We note that equations (91), (93) and (94) yield relation (31), which holds throughout the core. Equation (31) is indeed the original (15) with the σ^2 term dropped. By eliminating the constant E between (93) and (95), the latter takes on the form of the boundary condition (34). Longman uses (31) as one of the boundary conditions, but he leaves the yielding inside the core as undetermined and claims that no solution is possible when the core is not in neutral equilibrium.

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In the case of stable stratification, with $\beta < 0$, $\nu(s)$ in (90) becomes imaginary, and the function $M(s)$ in (91) to (95) is to be replaced by $\sin\left(\int_0^s |\nu| \mathrm{d}s\right)$.

The important feature of the asymptotic equations (91) to (95) is that the divergence X has only a term of the boundary-layer type. In the case of unstable stratifications, when $\nu(s)$ is real, $M(s)$ goes down exponentially with depth below the core surface, while for stable stratifications, when $\nu(s)$ becomes imaginary, X oscillates with a wavelength which is proportional to σ.

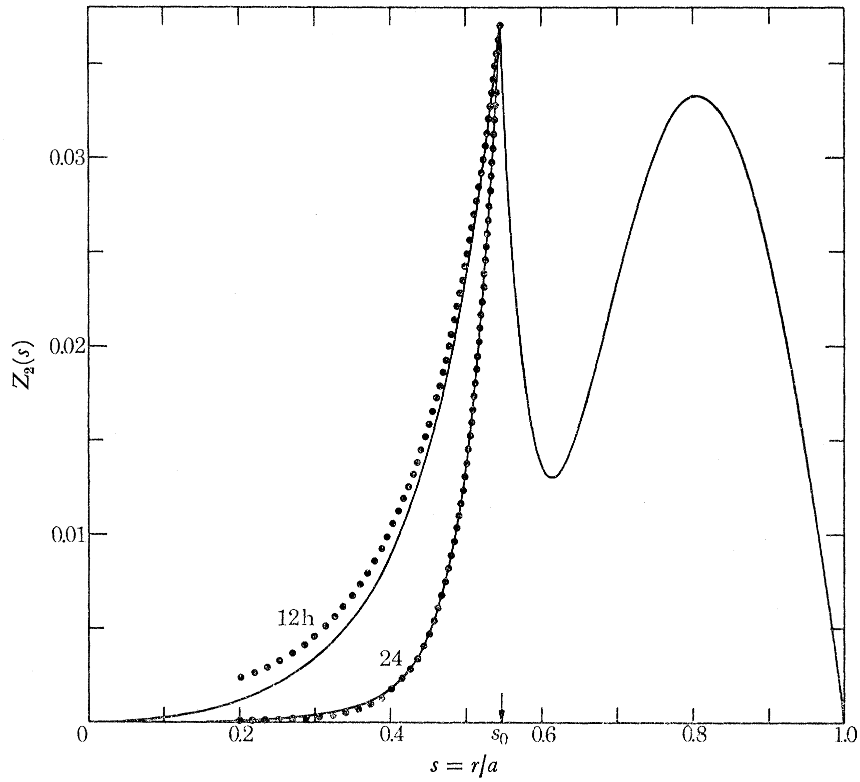

FIGURE 3. The stress function Z_2 for a uniformly unstable liquid core model with $\beta = 0.1$. —, Exact solution; •••, asymptotic solution. $n = 2$.

5. Discussion of results

We have determined the Love numbers for both stable and unstable uniform polytropic liquid core models, using, on the one hand, the exact equations (5) to (7) and, on the other hand, the approximate asymptotic equations (87) to (85). The mantle, in all cases, is as in model M_3 shown in table 2. The stress function Z_2 is shown in figures 3 and 4 for the unstable models with $\beta = 0.1$ and $\beta = 0.2$ respectively. In the case of $\beta = 0.1$, shown in figure 3, the asymptotic solution, given by the dots, is distinguishable from the exact solution at a period of 12 h, but nearly coincides with the exact solution at higher periods. For the case $\beta = 0.2$, shown in figure 4, the asymptotic approximation is even better. It is clear that, as predicted, with increasing period the stress function Z_2 tends to vanish throughout the core, except for a boundary layer near the core's surface.

The distribution of the stress function Z_2 is quite different in a uniformly stable model, as is shown in figure 5. Here Z_2 oscillates within the core, with an amplitude which increases toward

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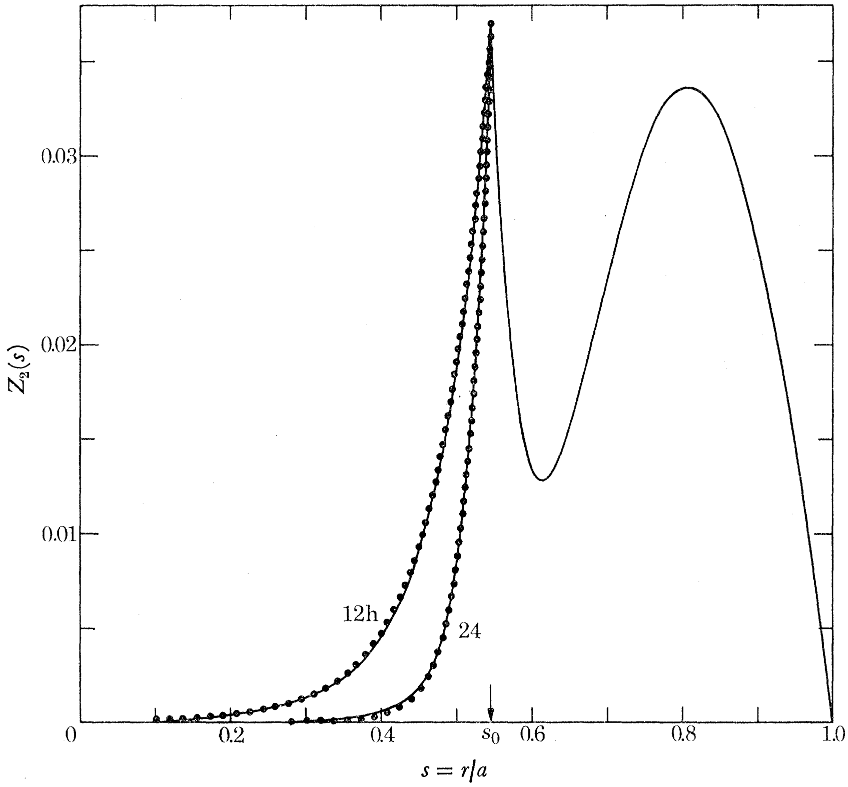

FIGURE 4. The stress function Z_2 for a uniformly unstable liquid core model with $\beta = 0.2$. —, Exact solution; •••, asymptotic solution. $n = 2$.

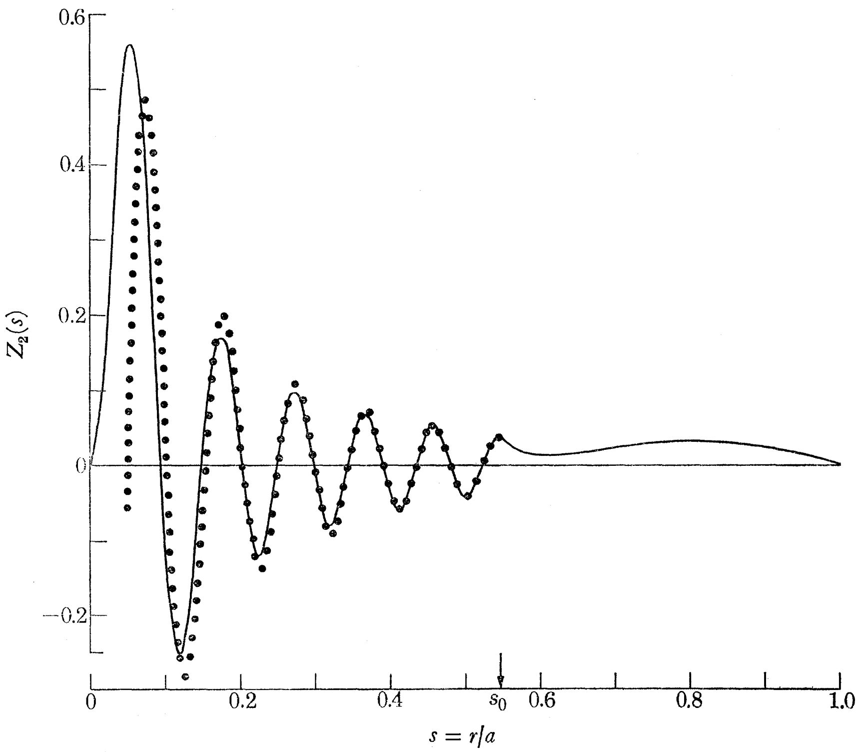

FIGURE 5. The stress function Z_2 for a uniformly stable liquid core model with $\beta = -0.2$. $T = 47$h. —, Exact solution; •••, asymptotic solution. $n = 2$.

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TABLE 4. VALUES OF THE LOVE NUMBERS h, k, l FOR THE LIQUID CORE POLYTROPIC MODELS WITH $\beta = -0.2$, 0 AND 0.2

The values for $T = \infty$ were derived from the asymptotic solution.

T/h	h			k			l		
	$\beta = -0.2$	0	0.2	-0.2	0	0.2	-0.2	0	0.2
6	0.6243	0.6253	0.6256	0.3068	0.3076	0.3084	0.0847	0.0847	0.0848
12	0.6162	0.6152	0.6154	0.3022	0.3028	0.3035	0.0840	0.0843	0.0843
24	0.6120	0.6127	0.6128	0.3009	0.3016	0.3023	0.0841	0.0842	0.0842
∞	0.6118	0.6119	0.6119	0.3005	0.3012	0.3019	0.0840	0.0841	0.0842

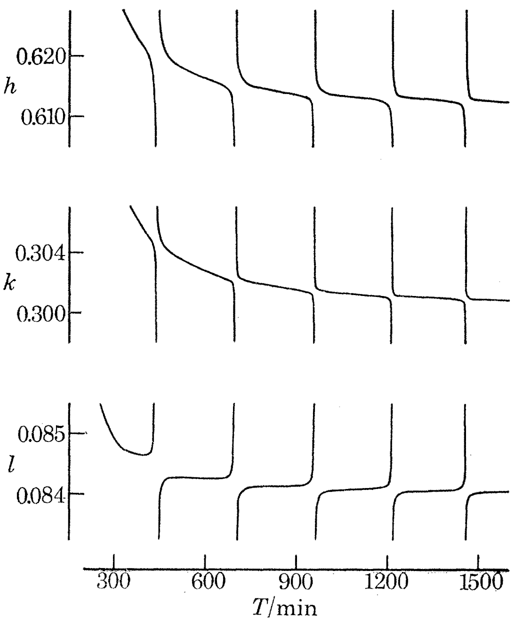

FIGURE 6. The Love numbers h, k, l as functions of the period T for a uniformly stable liquid core model with $\beta = -0.2$. $n = 2$.

the centre. The tendency for the wavelength of oscillation to decrease with increasing period is manifest.

The asymptotic solution given by the dots agrees well with the exact solution. It is to be noted that the scale of figure 5 is much larger than of figure 4; indeed, within the mantle the stress function Z_2 is identical in figures 4 and 5 to better than a few per cent. The Love numbers for the polytropic models are given in table 4.

Figure 6 shows the Love numbers h, k, l as functions of the period up to 25 h. It is seen that resonances occur at the periods of free oscillation which have a nearly constant spacing of about 4 h. The spacing is determined by the condition

$$\int_0^{s_0} |\nu(s)| \, \mathrm{d}s = \pi,$$

where $\nu(s)$ is defined in (90). In the case of the unstable models, the Love numbers vary in a monotone fashion, and very closely as represented by equation (1), as shown in figure 7. Indeed, the h, k, l curves in figure 7 are close to the curves in figure 6 except for the neighbourhoods of the

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resonances. The fit of the Love numbers to the quadratic approximation (1) is due to the dominance of the solution in the mantle.

The free oscillations for periods greater than the fundamental spheroidal mode of about 53.7 min, which we found for the stably stratified models, are 'core oscillations' (Pekeris, Alterman & Jarosch 1963), in the sense that their amplitude is confined primarily to the interior of the core. This is shown by the plot of the function $U(r)$ given in figure 8. It would be of interest to explore the mechanical effects of the rapid drop in stress within the boundary layer at the top of the core. It would also be of interest to obtain an accurate solution of the passage of an earthquake-pulse in the boundary layer.

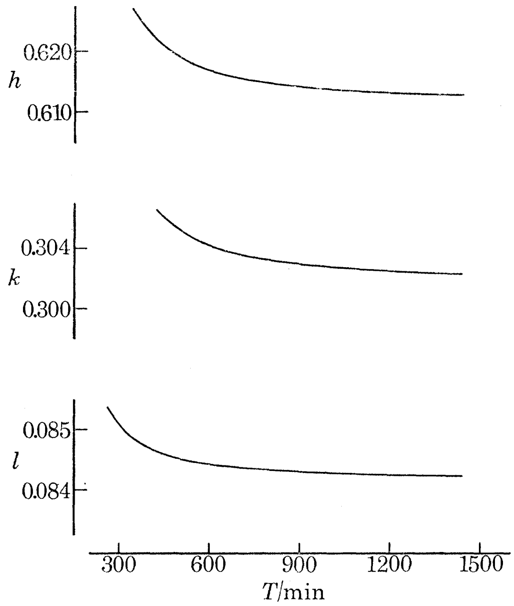

FIGURE. 7. The Love numbers h, k, l as functions of the period T for a uniformly unstable liquid core model with $\beta = 0.2$. $n = 2$.

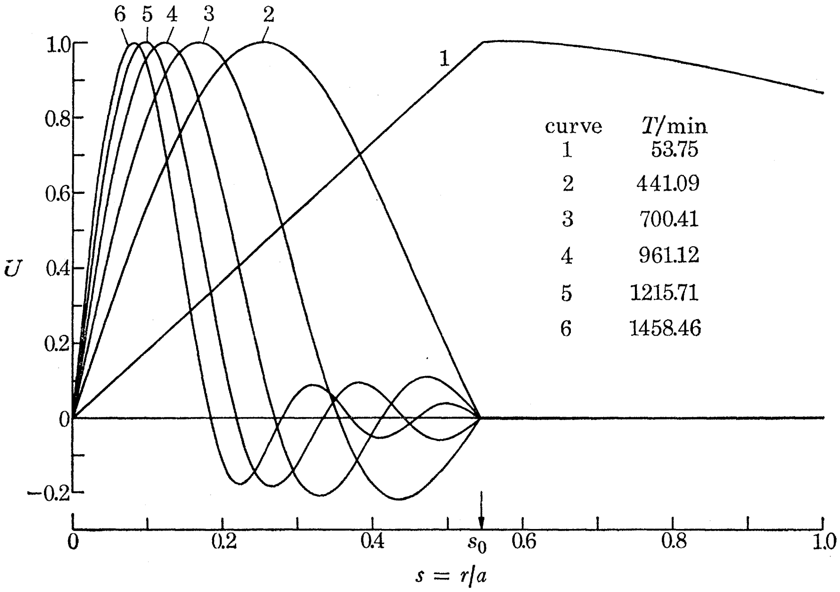

FIGURE 8. The radial displacement U for the free spheroidal oscillations for $n = 2$ in a uniformly stable liquid core model with $\beta = -0.2$. $n = 2$.

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6. The case of a liquid core in neutral equilibrium

The asymptotic expansion developed in § 4 proceeds in inverse powers of the large parameter ν, defined in (90), or in powers of $\sigma/\beta^{\frac{1}{2}}$. This asymptotic expansion is not valid in the limit of neutral equilibrium when $\beta = 0$. Writing (17) in the form

$$\beta g_0 X = \sigma^2(V + r\dot{V} - U) \equiv -\sigma^2 T, \tag{96}$$

we must have in the case $\beta = 0$ and σ finite

$$U = r\dot{V} + V. \tag{97}$$

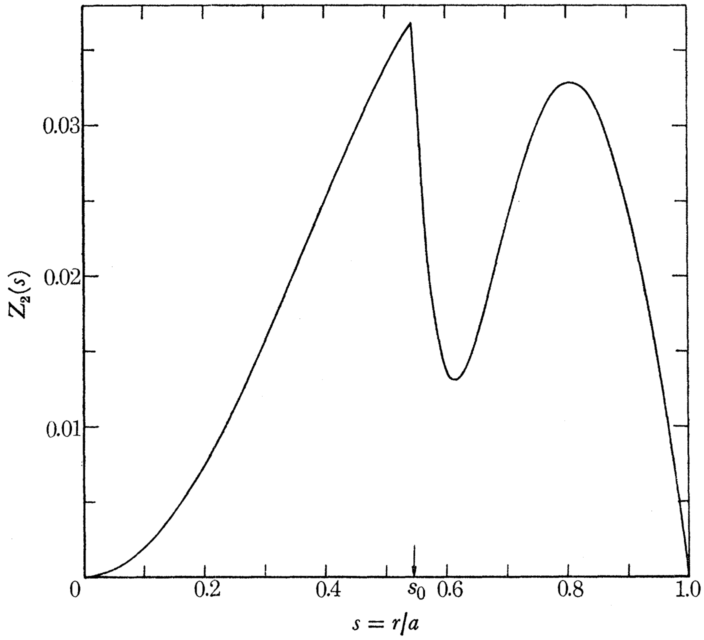

Figure 9. The stress function Z_2 for the case of a liquid core in neutral equilibrium ($\beta = 0$), for periods of 24 and 48 h. $n = 2$.

Figure 9 shows the stress function Z_2 for this model. We note that not only in the mantle but also in the core is the variation of Z_2 with period hardly distinguishable. Whatever variation does exist for $Z_2(s)$ in the range of periods of 12 to 48 h can be represented fairly accurately by a perturbation expansion

$$Z_2(s) = Z_2^{(0)}(s) + \sigma^2 Z_2^{(2)}(s) + \dots, \tag{98}$$

the second term being everywhere less than 3×10^{-5}.

In the limit of $\sigma = 0$ (and $\beta = 0$) the condition (97) is no longer necessary, but we would expect it to persist by continuity. We have solved equations (15) (with the σ^2-term dropped), (7) and (97), and the resulting stress function came out close to $Z_2^{(0)}(s)$ which was deduced from the solution for 24 and 48 h. This indicates that in the static limit of $\sigma = 0$, condition (97) persists in a liquid core in neutral equilibrium. Equation (97) provides the missing relation needed in order to make the yielding inside the core determinate. The resulting solution is illustrated in figure 10. Our conclusion is therefore that a density stratification in neutral equilibrium is not mandatory for a liquid core, but when neutral equilibrium happens to exist, the yielding in the static limit is determinate, and not arbitrary.

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If ω denotes the curl of the tidal displacement (u, v, w), then it follows from (2) that

$$\omega_r = 0, \quad \omega_\theta = \frac{T}{r\sin\theta}\frac{\partial S_n}{\partial\phi}, \quad \omega_\phi = -\frac{T}{r}\frac{\partial S_n}{\partial\theta}, \tag{99}$$

where
$$T = U - r\dot{V} - V. \tag{100}$$

In the case of neutral equilibrium, therefore, when the factor T vanishes (equation 97), the tidal displacement becomes *irrotational*.

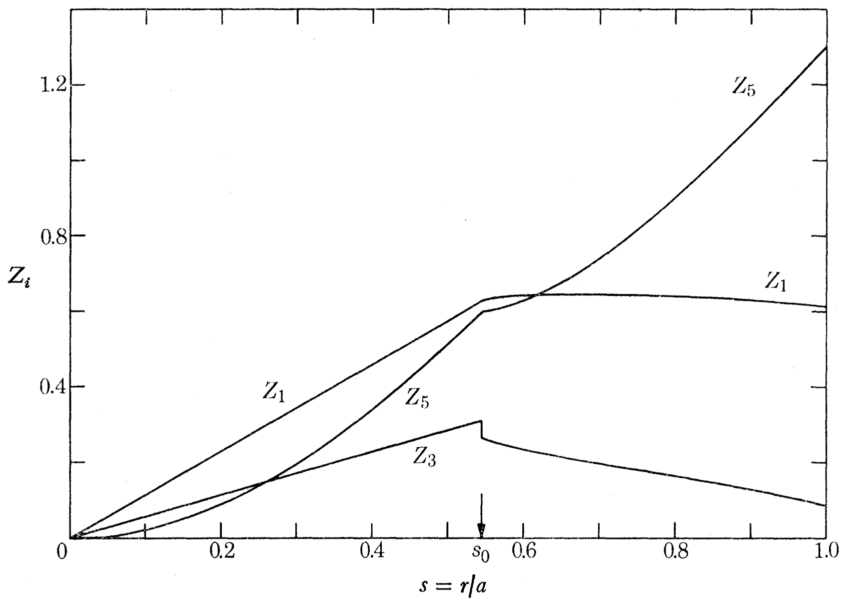

FIGURE 10. The radial displacement Z_1, the tangential displacement Z_3 and the perturbation in gravity Z_5 for a liquid core in neutral equilibrium ($\beta = 0$). $n = 2$. $U = aZ_1$; $V = aZ_3$; $P = ag_0(a)Z_5$.

7. Effect of a rigid inner core

We have treated a model in which the material for $0 < r < 1250$ km has a small rigidity $\mu = 0.5 \times 10^{12}(1-\alpha r^2)$ dyn/cm^2. For the case $\beta = 0.2$, shown in figure 11, the stress function Z_2 exhibits the rapid drop below the core boundary which we found to be characteristic for uniformly unstable liquid core models. In the case of stable stratification, the expected oscillatory behaviour of Z_2 in the liquid core, due to the existence of 'core oscillations', is shown in figure 12. The Love numbers for these hard-core models are very close to those given in table 4 for the polytropic models.

8. Summary

The dynamical response of the liquid core of the Earth to tidal forces depends on whether the density stratification in the liquid core is stable ($\beta < 0$), unstable ($\beta > 0$), or one of neutral equilibrium, where $\beta(r)$ is the stability parameter defined in (13).

In the case of neutral equilibrium, the tidal displacements are irrotational (equations 97 and 99). The stress and displacements vary in a regular manner, as shown in figures 9 and 10. The variation of the dynamical variables with the frequency σ of the tidal potential is slight, and is represented

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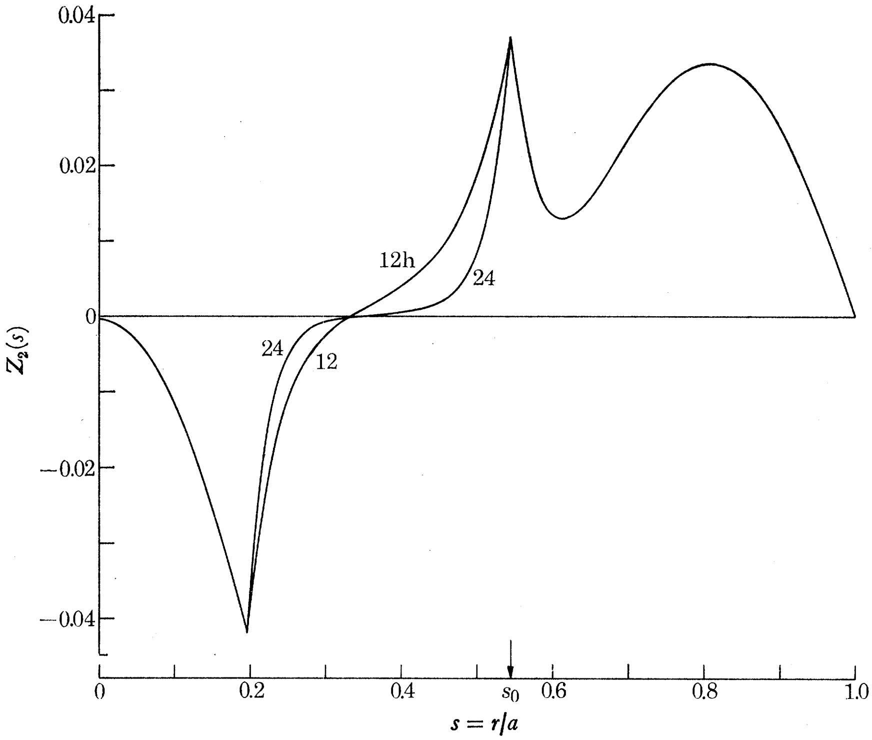

FIGURE 11. The stress function Z_2 for a model having an inner core ($r < 1250$ km, $s < 0.196$) with rigidity $\mu = 0.5 \times 10^{12}(1-\alpha r^2)$ dyn/cm², enclosed by a liquid core with $\beta = 0.2$. $n = 2$.

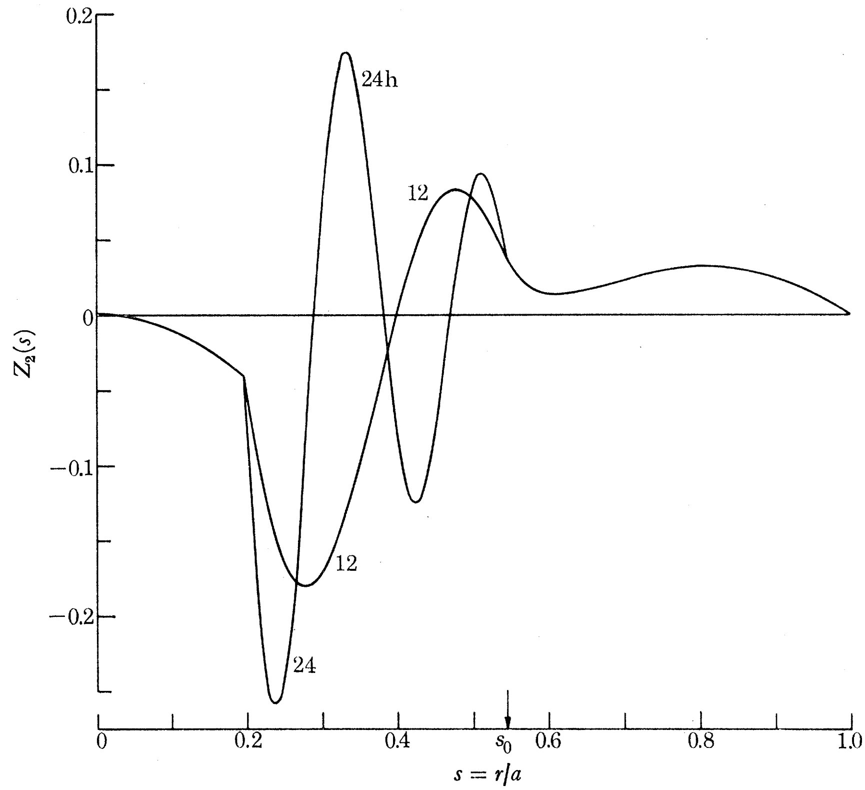

FIGURE 12. The stress function Z_2 for a model having an inner core ($r < 1250$ km, $s < 0.196$) with rigidity $\mu = 0.5 \times 10^{12}$ $(1-\alpha r^2)$ dyn/cm², enclosed by a liquid core with $\beta = -0.2$. $n = 2$.

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well by the quadratic form (98), in the range of tidal periods of 12 h to ∞. The vanishing of vorticity (97) supplies the missing condition needed in order to make the distribution of dynamical variables inside the core determinate. The Love numbers h, k, l vary quadratically with frequency σ (equation 1), and are shown in table 4. No free oscillations exist with periods greater than about 53.7 min corresponding to the spheroidal oscillation with $n = 2$.

In the case of an unstable density stratification ($\beta > 0$) the vorticity factor T in the controlling equation (96) does not vanish, and the motion is rotational. As $\sigma \to 0$ the divergence X of the tidal displacements tends to zero through most of the core, except for a boundary layer near the core surface whose thickness decreases with decreasing σ. Within the boundary layer the stress rises from a near-zero value to a finite value at the bottom of the mantle, as shown in figures 1, 3 and 4.

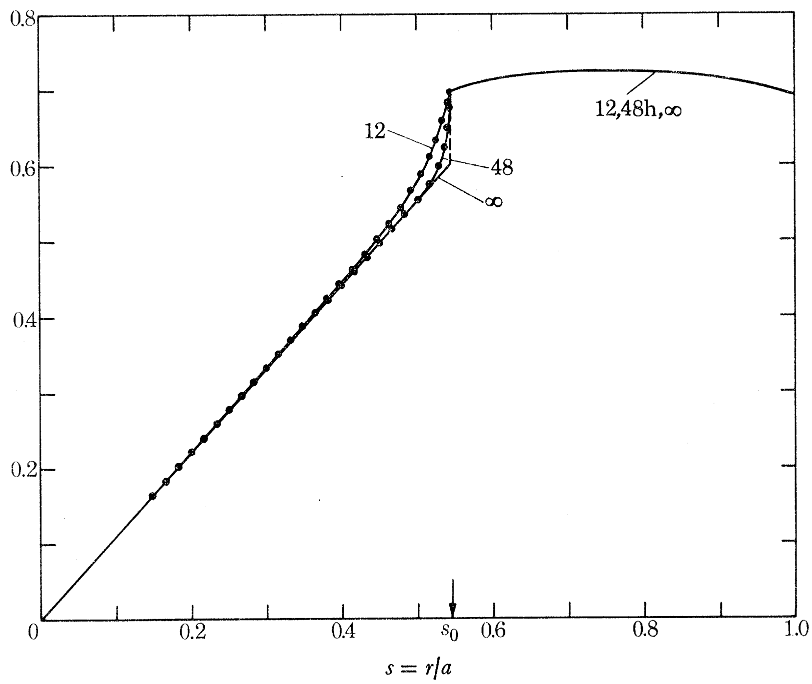

FIGURE 13. The radial displacement $Z_1 = U/a$ for the 'α-model' for periods of 12 and 48 h. —, Exact solution; •••, asymptotic solution. $n = 2$. See figure 1.

In the limit of $\sigma = 0$ the stress becomes discontinuous at the core surface. The radial displacement U also tends to become discontinuous as $\sigma \to 0$, as is shown in figures 13 and 14. The discontinuity in U is from a finite value inside the core surface to another finite value at the bottom of the mantle. Within the boundary layer the dynamical variables fall exponentially with depth d below the surface of the core as $\exp(-d/L)$, where L is proportional to the frequency. For the fortnightly tide the exponent becomes unity at a depth d of the order of several km. The Love numbers vary with the period of the tidal potential nearly quadratically as represented by equation (1), as is shown in figure 7. As in the case of neutral equilibrium, there are no free oscillations with periods greater than about 53.7 min.

The rapid variation of stress and of the radial displacement across the liquid boundary layer near the surface of the core is likely to upset further the nascent instability due to the assumed unstable density stratification. We may expect that convection would be initiated and penetrate downward from the boundary layer under the action of the long-period tidal forces, and that the

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resulting mixing would tend to establish a state of neutral equlibrium at the top layers of the core. The tendency toward a discontinuity in the radial displacement in the liquid boundary layer in the static limit may even result in mechanical deterioration of the mantle layer contiguous to the core.

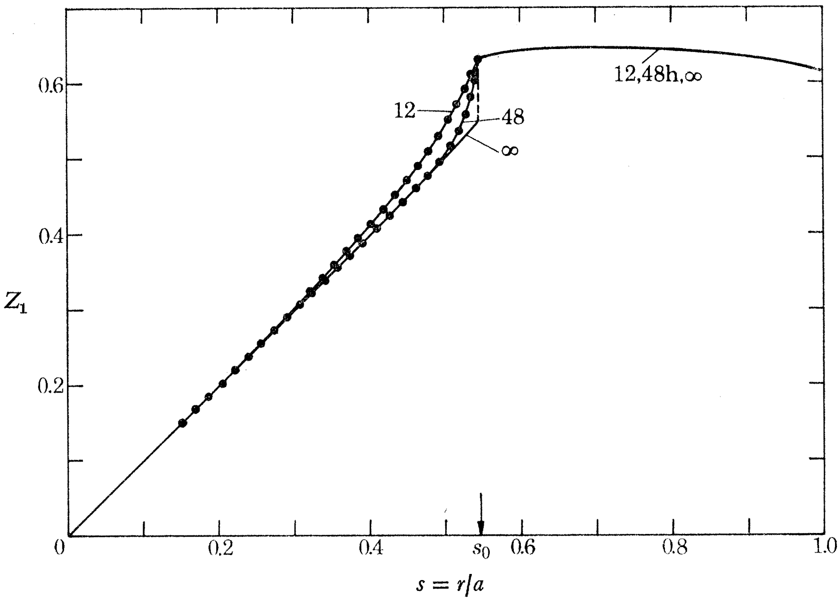

FIGURE 14. The radial displacement $Z_1 = U/a$ for a uniformly unstable core model with $\beta = 0.2$ for periods of 12 and 48 h. —, Exact solution; •••, asymptotic solution. $n = 2$. See figure 4.

In the case of a stable density stratification in the liquid core ($\beta < 0$) the dynamical response of the core to tidal disturbances is affected by the existence of an infinite number of free oscillations whose periods increase indefinitely. The amplitudes of these long-period free oscillations are confined to the core, as shown in figure 8; hence their designation as core oscillations. The Love numbers are shown in figure 6, exhibiting resonance at the periods of the free core oscillations. The periods of the core oscillations are separated by a nearly constant interval determined from

$$\int_0^{s_0} |\nu(s)|\, ds = \pi, \tag{101}$$

where $\nu(s)$ is defined in (90). For large N the periods T_N of the core oscillations are given asymptotically by

$$\int_0^{s_0} |\nu(s)|\, ds = N\pi. \tag{102}$$

The motion is rotational, and the controlling equation (96) is satisfied by

$$\beta g_0 X/T = -\sigma^2 \to 0, \tag{103}$$

with X and T both having oscillatory distributions inside the core. This is exemplified by the curve $Z_2 = \lambda X$ shown in figure 5. The period of 47 hours was chosen as lying midway between resonances. Since the amplitude of X does not tend to decrease with decreasing σ (figure 5), it follows from (103) that the amplitude of the vorticity factor T grows as $\sigma \to 0$. This feature, which is bound up with the existence of the core oscillations is likely to be affected by frictional forces. In any case this is not the first time we are faced with an 'infrared catastrophe'.

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